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POST-BUCKLING ANALYSIS OF AN ELASTICALLY-RESTRAINED COLUMN†

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Abstract—The nonlinear problem of the asymmetric snap-through buckling of a cantilever column that is restrained at its tip by a stiff, inclined wire, and is loaded laterally by a tip force, admits an exact solution which was determined previously. The structure was found to be imperfection-sensitive if one considers the combined extensional stiffnesses of the wire and of the column centerline to play the role of an imperfection. The same problem is now solved using the general Koiter method of analysis for the near post-buckling equilibrium. This present result for the post-buckling load vs. deflection relation for the "perfect structure" (infinite extensional stiffnesses) is shown to be an asymptotic representation of the corresponding exact result for vanishingly small deflection. At positive deflection, the approximate values for load in the asymptotic representation are less than the exact values. A similar conclusion is drawn for the buckling load vs. imperfection amplitude relation for the imperfect structure (finite extensional stiffnesses).

NOTATION

A	cross-sectional area
A_2, A_3	defined by equations (35c, d)
a	length of undeformed wire
a^*	length of deformed wire
B_1	defined by equation (56b)
C	amplitude of the linear buckling mode
d	end-shortening of column, equation (3)
E	Young's modulus
e	linear strain of column centerline, equation (7b)
e_w	strain in wire, equation (2)
e_{w1}	$\equiv e_w \sin^2\theta$, equation (7a)
F	lateral force applied to column
F_2	defined by equation (20d)
f	nondimensional applied force to column, equation (8a)
I	moment of inertia
J, K	nondimensional Lagrange multipliers, equations (8b, c)
$K(y)$	defined by equation (16)
l	length of undeformed centerline of column
P	axial compressive force on column
$P_2(y), P_3(y)$	second and third order terms in the potential energy
T	tensile force in wire
$U(X)$	displacement of the column centerline in the X direction
$u(x)$	$\equiv U/l$
V	total potential energy
$\bar{V}$	defined by equation (4)
V_1	nondimensional potential energy, equation (9)
V_1^*	potential energy with $K = K(y)$; equation (17a)

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X	coordinate along column centerline
x	$\equiv X/l$
$Y(X)$	lateral displacement of column centerline
$y(x)$	$\equiv Y/l$
$y_B(x)$	linear buckling solution for perfect structure, equation (28b)
z	nondimensional load parameter, equation (31)
α, β	Lagrange multiplier, equation (4)
δ	$\equiv y(1)\tan\theta \equiv Y_l/l \tan\theta$; tip deflection parameter
δV_1	first variation of V_1
ε	a small number
$\eta(x)$	admissible variation of $y(x)$
θ	angle between wire and horizontal in the undeformed state; see Fig. 1
θ^*	deformed value of θ
λ	axial compressive stiffness parameter of the column centerline, equation (42b)
μ	total elastic compliance parameter, equation (56a)
ν	elastic compliance parameter of the wire, equation (42a)
$\varphi(x)$	shape of linear buckling solution: $y_B = C\varphi$

Subscripts

c	classical value for buckling
cr	critical value for snap-through
w	wire
e	values for finite elastic compliance
$_1$	values at $x = 1$

1. INTRODUCTION

THE PROBLEM of buckling of a cantilever column restrained by a stiff wire inclined at an angle θ, and loaded by a lateral tip force F (see Fig. 1), was solved exactly in [1]. It was shown that the combined extensional stiffnesses of the column centerline and the restraining wire can be considered to act as imperfections to a "perfect" structure that has an inextensible centerline and is restrained by an inextensible wire. It was found that the snap-through buckling load of the "imperfect" structure is sensitive to such imperfections. The two extensional stiffnesses are combined in a nondimensional parameter μ. In [1], the problem was solved for finite stiffnesses ($\mu > 0$), and the solution for infinite stiffnesses was found by going to the limit ($\mu \to 0$).

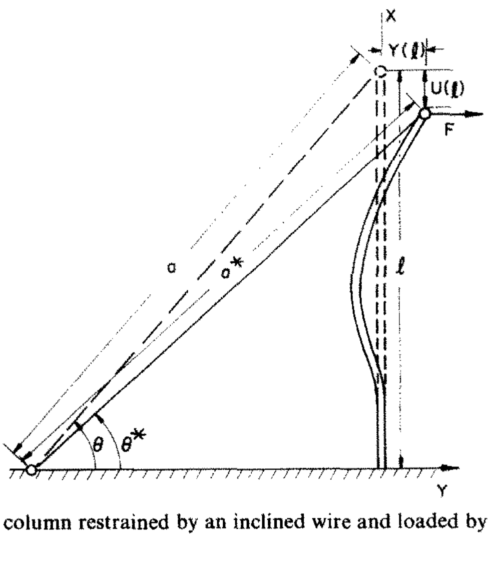

FIG. 1. Geometry of column restrained by an inclined wire and loaded by a lateral tip force F.

A general study of the local character of critical points in the theory of elastic stability, in particular with regard to post-buckling behavior and the effects of initial imperfections, has been developed by Koiter [2]. Further development of the theory for systems described by a finite number of generalized coordinates has been made recently by Thompson [3, 4] and by Roorda [5, 6]. In the more general terminology, as is used for example by Thompson [4], critical points of "perfect" structures are called branching points or points of bifurcation. There are three general types of branching points: (a) asymmetric; (b) stable symmetric; (c) unstable symmetric.

In the present problem for the inextensible centerline and wire, $\mu = 0$, the branching point is asymmetric; the problem treated in [1] and in this paper is a simple illustration of a continuous elastic structure that exhibits an asymmetric point of bifurcation. For μ positive but small compared to unity, the equilibrium paths in the load vs. deflection plane are close to those found for the inextensible wire. For the imperfect structure, $\mu > 0$, the critical points are of the snap-through type.

In this paper, the problem of Fig. 1 is solved using Koiter's method for post-buckling analysis as, for example, [7]. In this method, the buckling mode for the perfect structure is found by making an energy functional stationary subject to the nonlinear constraint of the wire. It was assumed in [1], for μ sufficiently small, that the linear bending approximation is adequate for representation of the energy. The geometrical nonlinearity, which is essential for the snap-through instability, arises in this problem from the nonlinear constraint imposed by the wire. The buckling mode is then substituted back into the energy functional in order to find the near post-buckling behavior and the imperfection sensitivity of the structure.

The results obtained by Koiter's method agree very well with results presented in [1]; these results give asymptotic representations of the exact solution when the post-buckling deflection or the imperfection tend to zero. This is verified in Appendices A and B, in which the asymptotic expressions for the post-buckling behavior and imperfection sensitivity of the structure are obtained from the exact expressions found in [1].

The specific methods developed for systems described by a finite number of generalized coordinates can also be applied to this problem and yield the same results. In Appendix C, the method of Roorda [6] is demonstrated.

These results offer a transparent example of the application of Koiter's method to a simple problem and, additionally, introduce two points that are perhaps novel. The first point is the use of the method for imperfections that are not geometric. The second point is that the method is applied to a case where the energy functional for the perfect structure is not found as the limit of the functional for the imperfect structure as μ tends to zero. This is the case where a nonlinear kinematic constraint, included in the functional by use of a Lagrange multiplier, necessitates as a first step the elimination of the multiplier before the energy functional can be written in the usual form.

2. ENERGY FUNCTIONAL FOR THE PERFECT STRUCTURE

For the perfect structure (Fig. 1), the total potential energy V is

$$V = \frac{1}{2}EI\int_0^l [\mathrm{d}^2 Y(X)/\mathrm{d}X^2]^2\,\mathrm{d}X - FY(l). \tag{1}$$

With the wire and column centerline both inextensible, the prebuckling displacements are zero. Buckling will occur when, for a nontrivial displacement $Y(X)$, V is stationary subject to the following two constraints: the strain e_w in the wire is zero for this displacement, where, as in [1],

$$e_w = l^{-1}\sin^2\theta\{Y(l)\cot\theta - U(l) + (2l)^{-1}[Y(l)]^2 + (2l)^{-1}[U(l)]^2\} = 0 \tag{2}$$

the end-shortening d of the column centerline is zero for this displacement, where

$$d = U(l) - \frac{1}{2}\int_0^l [\mathrm{d}Y(X)/\mathrm{d}X]^2\,\mathrm{d}X = 0. \tag{3}$$

With the introduction of the Lagrange multipliers α and β, we then seek to make the functional $\bar{V}$ stationary, where

$$\bar{V} \equiv V + \alpha e_w + \beta d. \tag{4}$$

With use of nondimensional variables

$$x \equiv X/l, \qquad y(x) \equiv Y/l, \qquad u(x) \equiv U/l \tag{5}$$

we can write, with primes denoting x-differentiation,

$$\bar{V} = \frac{EI}{2l}\left[\int_0^1 (y'')^2\,\mathrm{d}x - \frac{2Fl^2}{EI}y(1) + \frac{2\alpha l\sin^2\theta}{EI}e_{w1} + \frac{2\beta l^2}{EI}e\right] \tag{6}$$

where the strain parameters e_{w1} and e are defined as

$$e_{w1} \equiv y(1)\cot\theta - u(1) + \frac{1}{2}y(1)^2 + \frac{1}{2}u(1)^2 = e_w/\sin^2\theta \tag{7a}$$

$$e \equiv u(1) - \frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x. \tag{7b}$$

Equation (6) takes a more convenient form with the introduction of the following parameters:

$$f \equiv \frac{Fl^2}{EI}\tan\theta \tag{8a}$$

$$K \equiv \frac{\alpha l\sin^2\theta}{EI} \tag{8b}$$

$$J \equiv \frac{\beta l^2}{EI} \tag{8c}$$

$$V_1 \equiv \frac{2l\bar{V}}{EI}. \tag{8d}$$

The functional V_1 can then be expressed as

$$V_1 = \int_0^1 (y'')^2\,\mathrm{d}x - 2fy(1)\cot\theta + 2Ke_{w1} + 2Je. \tag{9}$$

Let $y(x)$ be a function for which V_1 is stationary, and let $\hat{y}(x) = y(x)+\varepsilon\eta(x)$, with $|\varepsilon| \ll 1$ and $\eta(x)$ an admissible variation. Then a necessary condition for stationary V_1,

$$\delta V_1 = 0 \tag{10a}$$

is

$$\frac{\partial}{\partial\varepsilon} V_1(y+\varepsilon\eta)|_{\varepsilon=0} = 0. \tag{10b}$$

Further conditions at $\varepsilon = 0$ are

$$\frac{\partial V_1}{\partial u(1)} = 0, \qquad \frac{\partial V_1}{\partial K} = 0, \qquad \frac{\partial V_1}{\partial J} = 0. \tag{10c, d, e}$$

Application of equation (10c) to equation (9), with use of equations (7a, b), gives

$$\frac{\partial V_1}{\partial u(1)} = 2K[-1+u(1)]+2J = 0. \tag{11}$$

However, the assumptions of linear bending theory and small strains, that have already been adopted, imply that, for consistency, we must set

$$1-u(1) \doteq 1. \tag{12}$$

Hence, in view of this, equation (11) gives

$$K = J. \tag{13}$$

Equation (10b) when applied to equation (9), together with equations (7a, b) and equation (13), gives the following variational equation to determine $y(x)$:

$$\int_0^1 (y''\eta''-Ky'\eta')\,\mathrm{d}x+\eta(1)\{K[y(1)+\cot\theta]-f\cot\theta\} = 0. \tag{14}$$

At $x = 0$ we assume clamped conditions, while at $x = 1$, equation (14) gives the natural boundary conditions. From equation (14) we get the Euler–Lagrange differential equation for $y(x)$,

$$y''''+Ky'' = 0, \qquad 0 < x < 1 \tag{15a}$$

and the following four boundary conditions:

$$y = y' = 0 \qquad \text{at } x = 0 \tag{15b, c}$$

$$\left.\begin{aligned} &y'' = 0 \\ &y'''+f\cot\theta-K(y+\cot\theta-y') = 0 \end{aligned}\right\} \text{at } x = 1. \tag{15d, e}$$

In addition, the two constraints

$$e_{w1} = y(1)\cot\theta-u(1)+\frac{1}{2}y(1)^2+\frac{1}{2}u(1)^2 = 0 \tag{15f}$$

$$e = u(1)-\frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x = 0 \tag{15g}$$

are to be satisfied; this also follows from equations (10d, e).

3. BUCKLING AND POST-BUCKLING OF THE PERFECT STRUCTURE

The exact solution satisfying equations (15a–g) was found in Section 4 of [1]; there it was determined using the limiting case of infinite stiffnesses for an elastic wire and an elastic column centerline. To use the Koiter technique in the present problem, in which infinite stiffnesses (zero compliances) have already been assumed, it is necessary to eliminate the Lagrange multipliers K and J from the expression for the energy functional, equation (9). For this, we use the equilibrium conditions, equations (13) and (15d), to write

$$K = J = K(y) \equiv \frac{y'''(1)+f\cot\theta}{y(1)+\cot\theta-y'(1)}. \tag{16}$$

With this substitution into equation (9), the functional $V_1^*(y)$ is formed,

$$V_1^*(y) = \int_0^1 (y'')^2\,\mathrm{d}x - 2fy(1)\cot\theta + 2K(y)(e_{w1}+e) \tag{17a}$$

where, with use of equation (15g),

$$e_{w1}+e = y(1)\cot\theta + \frac{1}{2}y(1)^2 - \frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x + \frac{1}{2}\left[\frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x\right]^2. \tag{17b}$$

It is easily verified that the functionals $V_1^*(y)$ and $V_1(y)$ have the same extremum $y(x)$, which is the solution to equations (15a–g). Hence, in place of equation (10a), we seek to make

$$\delta V_1^*(y) = 0. \tag{18}$$

For y and y' sufficiently small, $K(y)$ can be represented by a power series, and the functional $V^*(y)$ can then be rearranged, grouping together terms of equal degree in y or y-derivatives, i.e.

$$V_1^*(y) = P_2(y)+P_3(y)+\ldots. \tag{19}$$

We write

$$\begin{aligned}[\cot\theta+y(1)-y'(1)]^{-1} &= (\cot\theta)^{-1}\left[1+\frac{y_1-y_1'}{\cot\theta}\right]^{-1}\\ &= \tan\theta\{1-[y(1)-y'(1)]\tan\theta\\ &\quad+[y(1)-y'(1)]^2\tan^2\theta-\ldots\}\end{aligned}$$

and substitute into equation (16). Then, if the $Q_i(y)$ are polynomials, homogeneous of degree i, in y and y-derivatives, we can write

$$K(y)(e_{w1}+e) = Q_1(y)+Q_2(y)+Q_3(y)+\ldots \tag{20a}$$

where [setting $y(1) = y_1$, $y'(1) = y'_1$, etc.],

$$Q_1(y) = fy_1 \cot\theta \tag{20b}$$

$$Q_2(y) = y_1[y_1''' - f(y_1 - y_1')] + \frac{1}{2}f\left[y_1^2 - \int_0^1 (y')^2\,\mathrm{d}x\right] \tag{20c}$$

$$Q_3(y) = -\frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x(y_1''' + fy_1')\tan\theta + y_1 F_2(y_1, y_1', y_1'''). \tag{20d}$$

In equation (20d), F_2 is a homogeneous polynomial, of second degree in its arguments, whose further expression is not necessary. Substitution from equations (20) into equations (17a, b) and comparison with equation (19) gives

$$P_2(y) = \int_0^1 (y'')^2\,\mathrm{d}x + 2Q_2(y) = \int_0^1 [(y'')^2 - f(y')^2]\,\mathrm{d}x + y_1[2y_1''' + f(2y_1' - y_1)] \tag{21}$$

$$P_3(y) = 2Q_3(y) = -\tan\theta(y_1''' + fy_1')\int_0^1 (y')^2\,\mathrm{d}x + 2y_1 F_2. \tag{22}$$

The buckling load and buckling modes are then found from the stationary condition {[10], equation (3.5.3)},

$$\delta P_2(y) = 0 \tag{23a}$$

together with the constraint, equation (15f), which is reduced to lowest degree for small y, viz.

$$y_1 = 0. \tag{23b}$$

Equations (21) and (23a, b) give the following differential equation

$$y'''' + fy'' = 0, \qquad 0 < x < 1 \tag{24a}$$

and the boundary conditions

$$y = y' = 0 \qquad \text{at } x = 0 \tag{24b, c}$$

$$y'' = y = 0 \qquad \text{at } x = 1. \tag{24d, e}$$

The solution of equation (24a) that also satisfies equations (24b, c, d) is

$$y(x) = C\{\sin\sqrt{f} - \sin[\sqrt{f}(1-x)] - \sqrt{f}x\cos\sqrt{f}\}. \tag{25}$$

Condition (24e) becomes, upon substitution from equation (25),

$$\sin\sqrt{f} - \sqrt{f}\cos\sqrt{f} = 0. \tag{26}$$

The smallest positive root of equation (26) is $f = f_c$,

$$f_c = 20{\cdot}19 \tag{27}$$

which is the nondimensional buckling load. The buckling solution $y = y_B(x)$ and buckling mode $\varphi(x)$ are

$$\varphi(x) = \sin\sqrt{f_c} - \sin[\sqrt{f_c}(1-x)] - \sqrt{f_c}x\cos\sqrt{f_c} \tag{28a}$$

$$y_B(x) \equiv C\varphi(x). \tag{28b}$$

The next approximation to equation (18) for small-deflection post-buckling representation is

$$\delta[P_2(y)+P_3(y)] = 0. \tag{29a}$$

The solution to equation (29a) must also satisfy the constraints, equation (15f, g), to the same order of approximation, viz.

$$y_1\cot\theta - \frac{1}{2}\int_0^1 (y')^2\,\mathrm{d}x + \frac{1}{2}y_1^2 = 0. \tag{29b}$$

In Koiter's method, the variational equation (29a) is approximated by the condition

$$\frac{\mathrm{d}}{\mathrm{d}C}[P_2(y_B)+P_3(y_B)] = 0 \tag{30}$$

which is asymptotically valid as the amplitude C and the parameter z,

$$z \equiv 1 - \frac{f}{f_c} \tag{31}$$

are each small compared to one. The square bracketed term in equation (30) is evaluated with the use of equations (28a, b) and the repeated use of equation (26) as an identity in f_c. Since $\varphi(1) = 0$, we obtain from equations (21) and (22),

$$P_2(C\varphi) = C^2\int_0^1 [(\varphi'')^2 - f(\varphi')^2]\,\mathrm{d}x \tag{32a}$$

$$P_3(C\varphi) = -\tan\theta C^3(\varphi_1''' + f\varphi_1')\int_0^1 (\varphi')^2\,\mathrm{d}x. \tag{32b}$$

From equations (28a) and (26),

$$\int_0^1 (\varphi')^2\,\mathrm{d}x = \int_0^1 [\sqrt{f_c}\{\cos[\sqrt{f_c}(1-x)] - \cos\sqrt{f_c}\}]^2\,\mathrm{d}x = \frac{1}{2}f_c\sin^2\sqrt{f_c} \tag{33a}$$

$$\int_0^1 (\varphi'')^2\,\mathrm{d}x = \int_0^1 \{f_c\sin[\sqrt{f_c}(1-x)]\}^2\,\mathrm{d}x = \frac{1}{2}f_c^2\sin^2\sqrt{f_c} \tag{33b}$$

$$\varphi'(1) = \sqrt{f_c} - \sin\sqrt{f_c} \tag{33c}$$

$$\varphi'''(1) = -f_c^{\frac{3}{2}}. \tag{33d}$$

Therefore, from equations (32),

$$P_2(C\varphi) = \tfrac{1}{2}zC^2 f_c^2\sin^2\sqrt{f_c} \tag{34a}$$

$$P_3(C\varphi) = \tfrac{1}{2}C^3 f_c^2\tan\theta\sin^3\sqrt{f_c}. \tag{34b}$$

By writing equations (34a, b) equivalently as

$$P_2(y_B) \equiv A_2\left(1-\frac{f}{f_c}\right)C^2, \qquad P_3(y_B) \equiv A_3C^3 \tag{35a, b}$$

with

$$A_2 \equiv \tfrac{1}{2}f_c^2 \sin^2 \sqrt{f_c} \tag{35c}$$

$$A_3 \equiv \tfrac{1}{2}f_c^2 \tan\theta \sin^3 \sqrt{f_c} \tag{35d}$$

the post-buckling condition, equation (30), is written

$$\frac{\mathrm{d}}{\mathrm{d}C}\left[A_2\left(1-\frac{f}{f_c}\right)C^2+A_3C^3\right] = 0. \tag{36}$$

For $C \neq 0$, equation (36a) gives

$$C = -\frac{2}{3}\frac{A_2}{A_3}\left(1-\frac{f}{f_c}\right). \tag{37}$$

Since $A_2 > 0$ but $A_3 < 0$, because the angle $\sqrt{f_c}$ (rad.) is in the third quadrant, then C is positive for $0 < f < f_c$. The tip deflection parameter δ.

$$\delta = y(1)\tan\theta \tag{38}$$

as a function of C is obtained from equation (29b) by putting $y = y_B(x)$ in the integral, viz.

$$y_1^2+2y_1\cot\theta-\frac{1}{2}C^2\int_0^1 (\varphi')^2\,\mathrm{d}x = 0. \tag{39}$$

The root of equation (39) which vanishes for $C = 0$ is

$$y_1 = \tan\theta(\tfrac{1}{4}C^2 f_c \sin^2\sqrt{f_c})+O(C^4). \tag{40}$$

By combining equations (37), (38) and (40) to eliminate C, we obtain

$$\frac{f}{f_c} = 1-\frac{3\sqrt{\delta}}{\sqrt{f_c}} \tag{41}$$

4. IMPERFECTION SENSITIVITY

As shown in [1], the elastic extensional compliance of the wire and the elastic axial compressive compliance of the column play the role here of imperfection parameters, because, in the limit as these parameters tend towards zero, the solution for the elastic case approaches that for the inextensible case discussed in the preceding section. It will now be shown that the results from the inextensible case can be used to determine an asymptotically valid formula for the critical snap-through load f_{cr} in the elastic case.

Consider now the nondimensional form V_{ie} of the strain energy for the elastic case; the same nondimensional nomenclature is used as previously, with the additional definitions now of the wire compliance parameter v,

$$v^{-1} \equiv \frac{E_wA_wl^2\sin^3\theta}{EI} \tag{42a}$$

and the axial stiffness parameter λ for the column,

$$\lambda = \frac{EAl^2}{EI}. \tag{42b}$$

Then V_{ie} is a functional of $y(x)$ and $u(x)$ and is defined by

$$V_{ie} = \int_0^1 (y'')^2 \,\mathrm{d}x + \lambda \int_0^1 \left[u' - \frac{1}{2}(y')^2\right]^2 \mathrm{d}x + v^{-1} e_{w1}^2 - 2fy(1) \cot \theta. \tag{43}$$

From the Schwarz Inequality, and the condition that $u(0) = 0$, we find that

$$\int_0^1 \left[u' - \frac{1}{2}(y')^2\right]^2 \mathrm{d}x \geq \left\{\int_0^1 \left[u' - \frac{1}{2}(y')^2\right] \mathrm{d}x\right\}^2 = \left[u(1) - \frac{1}{2}\int_0^1 (y')^2 \,\mathrm{d}x\right]^2 \equiv e^2. \tag{44}$$

The functional V_{ie}^*, defined by

$$V_{ie}^* = \int_0^1 (y'')^2 \,\mathrm{d}x + \lambda e^2 + v^{-1} e_w^2 - 2fy(1) \cot \theta \tag{45}$$

bears the following relationship to V_{ie}, assuming that λ, v^{-1}, f and θ are the same for each:

1. For every admissible $y(x)$ and $u(x)$, $V_{ie} \geq V_{ie}^*$.
2. V_{ie} and V_{ie}^* have the same equilibrium states.
3. At an equilibrium state $u(x)$ and $y(x)$, $V_{ie} = V_{ie}^*$.

Conclusion (1) follows at once from inequality (44). The Euler–Lagrange equations for V_{ie}^* can be readily shown to be equal to those given in [1] for V_{ie}. Finally, the equilibrium conditions can be shown to lead [with $u(0) = 0$] to

$$u' - \tfrac{1}{2}(y')^2 = \text{const.} = e, \qquad 0 < x < 1 \tag{46}$$

whereupon it is seen that the strict equality holds in (44) for equilibrium. From these remarks, it is seen that if V_{ie} is positive definite in a neighborhood of the function space about the equilibrium solutions $u(x)$ and $y(x)$, then V_{ie}^* is positive definite also in the same neighborhood.

Now compare V_{ie}^* with V_1, equation (9). We know [4] that the nondimensional Lagrange multiplier K must be, in a dimensionless form, the force in the wire that imposes the constraint $e_{w1} = 0$. It also follows by continuity arguments that K is the limit of the force in the wire for the case of vanishing wire and column compliance, viz. for

$$v \to 0 \qquad \text{and} \qquad \lambda^{-1} \to 0.$$

This means the the nondimensional force $v^{-1} e_{w1}$ in the elastic wire must be given asymptotically, for small v and λ^{-1}, by

$$v^{-1} e_{w1} = K[1 + O(v, \lambda^{-1})]. \tag{47a}$$

Similarly, the nondimensional axial force λe in the column is given asymptotically, for small v and λ^{-1}, by

$$\lambda e = J[1 + O(v, \lambda^{-1})]. \tag{47b}$$

The physical meaning of equations (47a, b) is readily derived. From the form of equation (4), it is evident that α must be equal to the product of the force T in the wire by the length a of the wire, i.e.

$$\alpha = Ta = Tl/\sin\theta. \tag{48}$$

Therefore, from equation (8b),

$$K = \frac{l\sin^2\theta}{EI}\alpha = \frac{l^2\sin\theta}{EI}T. \tag{49a}$$

If the force in the elastic wire is called T_e, then $T_e = E_w A_w e_w$, and

$$v^{-1}e_{w1} = \frac{E_w A_w l^2 \sin^3\theta}{EI}\cdot\frac{e_w}{\sin^2\theta} = \frac{l^2\sin\theta}{EI}T_e. \tag{49b}$$

Upon substitution from equations (49a, b), equation (47a) takes the form

$$T_e = T[1+O(v,\lambda^{-1})]. \tag{49c}$$

Similarly, for the rigid centerline on the column, β must equal the axial compressive force P in the column; then, from equation (8c)

$$J = \frac{Pl^2}{EI}. \tag{50a}$$

On the other hand, the elastic column axial force P_e is given by $P_e = EAe$, and also

$$\lambda e = \frac{EAl^2}{EI}e = \frac{Pl^2}{EI}. \tag{50b}$$

Hence, we get a formula similar to equation (49c):

$$P_e = P[1+O(v,\lambda^{-1})]. \tag{50c}$$

In view of equations (47a) and (47b), and the condition† $K = J$, equation (45) becomes

$$V_{ie}^* = V_1^* - K(e+e_{w1})[1-O(v,\lambda^{-1})]. \tag{51}$$

But

$$e_{w1} = vK[1+O(v,\lambda^{-1})] \tag{52a}$$

$$e = \lambda^{-1}K[1+O(v,\lambda^{-1})]. \tag{52b}$$

Then

$$V_{ie}^* = V_1^* - (v+\lambda^{-1})K^2\{1-[O(v,\lambda^{-1})]^2\}. \tag{53}$$

For K^2 in equation (53) we substitute $[K(y)]^2$ from equation (16), and set $y = y_B$ with C to be determined. We find that

$$[K(y_B)]^2 = f_c^2(1-2C\tan\theta\sin\sqrt{f_c}) \tag{54}$$

+ higher order terms in z and C. For $z \neq 0$, the amplitude C has a nonzero limit as $(v+\lambda^{-1})$ tends to zero [i.e. the limit found in equation (37)]. Hence, only terms of lowest order in the small quantities C and z need be retained in $[K(y)]^2$ when equation (35) is combined with equation (54) to obtain the following asymptotic expression for the total potential

† See Appendix A of [1] for proof that $P_e = T_e\sin\theta$ generally for small strains.

energy in equation (53):

$$V_{ie}^{*} = A_2 z C^2 + A_3 C^3 + B_1 \mu C + (\nu + \lambda^{-1}) f_c^2 \tag{55}$$

where

$$\mu \equiv (\nu + \lambda^{-1}) \tan^2 \theta \tag{56a}$$

$$B_1 \equiv 2 f_c^2 \sin \sqrt{f_c} \cot \theta. \tag{56b}$$

The conditions for a stationary point $f = f_{cr}$, $C = C_{cr}$ on the load vs. deflection curve f vs. C follow from equation (55):

$$\frac{\partial V_{ie}^{*}}{\partial C} = 0 \Rightarrow 2A_2 z C + 3A_3 C^2 + B_1 \mu = 0 \tag{57a}$$

$$\frac{\partial^2 V_{ie}^{*}}{\partial C^2} = 0 \Rightarrow 2A_2 z + 6A_3 C = 0. \tag{57b}$$

The solution of equations (57a, b) that gives positive C_{cr} for $\mu > 0$ is

$$z_{cr} = +\frac{(3A_3 B_1 \mu)^{\frac{1}{2}}}{A_2} \tag{58a}$$

$$C_{cr} = -\frac{A_2}{3A_3} z_{cr}. \tag{58b}$$

The instability of equilibrium at f_{cr}, C_{cr} can be shown to follow directly from equations (57a, b), since, for $z = z_{cr}$ and arbitrary C, the total potential energy can be written as

$$V_{ie}^{*} = A_3 (C - C_{cr})^3 + \text{const.} \tag{59}$$

The cubic form indicates that the total potential energy is not a minimum at $C = C_{cr}$, and instability of equilibrium at f_{cr}, C_{cr} then follows immediately from the theorem of Četaev [11].

We determine f_{cr} by substitution into equation (36a) from equations (21b), (21d) and (34c); thus we get the equation for imperfection sensitivity of the snap-through load,

$$\frac{f_{cr}}{f_c} = 1 - 2\sqrt{3\mu}. \tag{60}$$

This is exactly the asymptotic expression of f_{cr}/f_c for $\mu \to 0$ that is derived in Appendix B below from the results of [1].

5. NUMERICAL RESULTS AND DISCUSSION

The results for the post-buckling equilibrium of the "perfect" structure, equation (41), are shown in Fig. 2 and compared to corresponding results from [1]. Figure 3 gives a similar comparison for the imperfection sensitivity of the snap-through load, equation (60), again with good agreement.

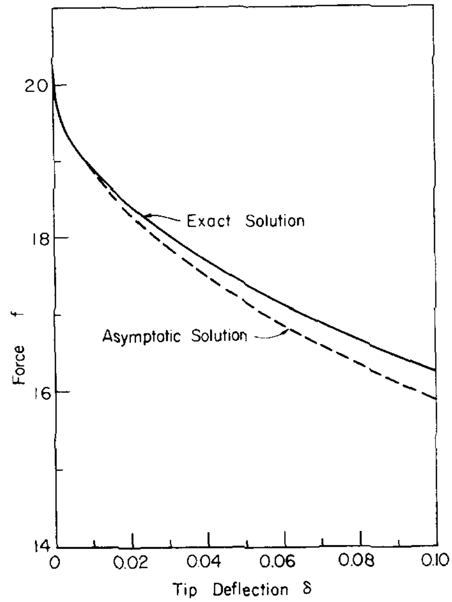

FIG. 2. Comparison between exact and asymptotic solutions for the post-buckling equilibrium of the "perfect" ($\mu = 0$) column.

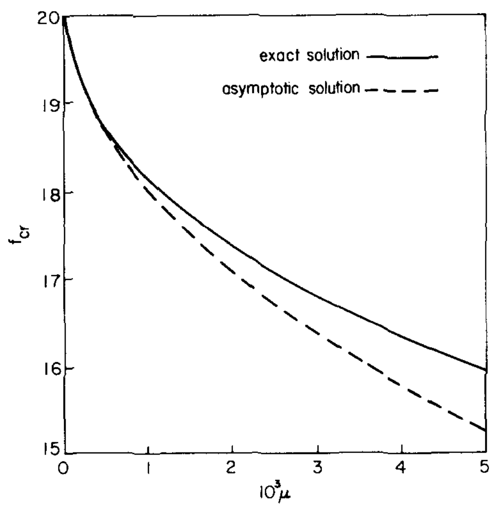

FIG. 3. Comparison between exact and asymptotic solutions for the buckling load f_{cr} of columns with elastic parameter $\mu > 0$.

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APPENDIX A

Post-buckling behavior from [1]

An asymptotic expression, valid for $\delta \to 0$, is derived from the exact solution for the post-buckling equilibrium load f vs. tip deflection δ as given parametrically in κ by two functions $\delta = \delta_+(\kappa)$, $f = f_+(\kappa)$ that are equations (28a) and (28b) of [1]:

$$\delta_+(\kappa) = 2\frac{K_1^2}{K_2} \tag{A1a}$$

$$f_+(\kappa) = \kappa^2\left(1+\frac{2K_1 \sin\kappa}{K_2}\right) \tag{A1b}$$

$$K_1(\kappa) \equiv \sin\kappa - \kappa\cos\kappa \tag{A1c}$$

$$K_2(\kappa) \equiv \tfrac{1}{2}\kappa^2 - \sin^2\kappa + \tfrac{1}{4}\kappa\sin 2\kappa. \tag{A1d}$$

Let κ_c be the critical value of κ, $[K_1(\kappa_c) = 0]$. For κ close to κ_c, we can assume that $\Delta \equiv \kappa_c - \kappa$ is also small, and we have

$$K_1(\kappa) \approx -\Delta\kappa_c^2\cos\kappa_c \tag{A2}$$

$$K_2(\kappa) \approx K_2(\kappa_c) = \tfrac{1}{2}\kappa_c^4\cos^2\kappa_c\,. \tag{A3}$$

So, asymptotic expressions are

$$\delta_+(\kappa) \approx 4\Delta^2 \tag{A4}$$

$$f_+(\kappa) = (\kappa_c-\Delta)^2(1-4\Delta) \approx \kappa_c^2 - 6\Delta\kappa_c. \tag{A5}$$

Substituting equation (A4) into equation (A5), we get

$$f_+(\kappa) = \kappa_c^2\left(1-\frac{3}{\kappa_c}\sqrt{\delta}\right). \tag{A6}$$

Since $\kappa_c^2 = f_c$, then equation (A6) becomes equation (41).

APPENDIX B

Imperfection sensitivity from [1]

For small positive values of the imperfection parameter μ, we seek to write

$$f_{cr} = f_c(1-B\mu^{\gamma}) \tag{B1}$$

where γ and B are constants. For $\mu > 0$, the *pre-buckling* equilibrium relation between δ and f is given parametrically in κ by two functions $\delta = \delta_-(\kappa), f = f_-(\kappa)$ that are equations (25) and (26) of [1]:

$$\delta_-(\kappa) = \frac{K_1^2}{K_2}\left[1-\left(1-2\mu\kappa^2\frac{K_2}{K_1^2}\right)^{\frac{1}{2}}\right] \tag{B2a}$$

$$f_-(\kappa) = \kappa^2\left[1+\frac{\delta_-(\kappa)\sin\kappa}{K_1}\right]. \tag{B2b}$$

Here, K_1 and K_2 are defined in equations (A1c, d) above. Upon combining equations (B2a, b), we get

$$f_-(\kappa) = \kappa^2\left\{1+\frac{\sin\kappa}{K_2}[K_1-(K_1^2-2\mu\kappa^2K_2)^{\frac{1}{2}}]\right\}. \tag{B3}$$

Since buckling occurs when $\mathrm{d}f/\mathrm{d}\delta = 0$, then the critical value of κ is the smallest positive root of the equation

$$\frac{\mathrm{d}f_-(\kappa_{cr})}{\mathrm{d}\kappa} = 0. \tag{B4}$$

The buckling load f_{cr} is therefore given by

$$f_{cr} = f_-(\kappa_{cr}). \tag{B5}$$

Equations (B4) and (B5) apply for $\mu > 0$; for $\mu = 0$, $\kappa_c = 4{\cdot}4938$ and

$$f_-(\kappa_c) = f_c = \kappa_c^2 = 20{\cdot}19. \tag{B6}$$

For small μ, we can assume that

$$\Delta \equiv \kappa_c - \kappa_{cr} \tag{B7}$$

is also small. Hence, [using the fact that $K_1(\kappa_c) = 0$],

$$K_1(\kappa_{cr}) = \sin\kappa_{cr} - \kappa_{cr}\cos\kappa_{cr} \approx -\Delta\kappa_c\sin\kappa_c = \Delta\kappa_c^2\cos\kappa_c \tag{B8a}$$

$$K_2(\kappa_{cr}) \approx K_2(\kappa_c) = \tfrac{1}{2}\kappa_c^2 - \sin^2\kappa_c + \tfrac{1}{4}\kappa_c\sin 2\kappa_c = \tfrac{1}{2}\kappa_c^4\cos^2\kappa_c. \tag{B8b}$$

Substituting equations (B8a, b) into equation (B3), we have

$$f_-(\kappa_{cr}) = f(\kappa_c - \Delta) \equiv f^*(\Delta)$$

and

$$\begin{aligned} f^*(\Delta) &\approx (\kappa_c-\Delta)^2\left\{1+\frac{2\sin\kappa_c}{\kappa_c^4\cos^2\kappa_c}[-\kappa_c^2\cos\kappa_c\Delta-(\kappa_c^4\cos^2\kappa_c\Delta^2-\mu\kappa_c^6\cos^2\kappa_c)^{\frac{1}{2}}]\right\} \\ &\approx (\kappa_c-\Delta)^2\left\{1+\frac{2}{\kappa_c}[-\Delta+(\Delta^2-\mu\kappa_c^2)^{\frac{1}{2}}]\right\} \\ &\approx \kappa_c^2 - 4\Delta\kappa_c + 2\kappa_c(\Delta^2-\mu\kappa_c^2)^{\frac{1}{2}}. \end{aligned} \tag{B9}$$

Equation (B4) is equivalent to

$$\frac{\mathrm{d}f^*(\Delta)}{\mathrm{d}\Delta} = 0 \Rightarrow -4\kappa_c + \frac{2\kappa_c\Delta}{(\Delta^2-\mu\kappa_c^2)^{\frac{1}{2}}} = 0. \tag{B10}$$

Since $\Delta > 0$, equation (B10) implies

$$\Delta = \tfrac{2}{3}\kappa_c\sqrt{3\mu}.$$

Substituting back into equation (B9), we obtain

$$f_{cr} = \kappa_c^2(1-2\sqrt{3\mu}). \tag{B11}$$

This expression and equation (A7) previous are asymptotic, of course, because in their derivation terms of order Δ^2 were neglected in comparison to terms of order Δ.

APPENDIX C

Results obtained by use of Roorda's method [6]

Let the amplitude C of the linear buckling mode shape, equation (20), be taken as the single principal generalized coordinate u_1 in accordance with the development and notation in [6]. The equilibrium equation in the notation of equation (10) of [6] is:

$$\frac{\partial}{\partial u_1}(s_{p_0+p}) = S_{u_1\varepsilon}\varepsilon+\frac{1}{2!}(S_{u_1u_1u_1}u_1^2+2S_{pu_1u_1}pu_1) = 0. \tag{C1}$$

The equivalence of this equation to equation (57a) in our paper is seen by the following correlation of nomenclature (our nomenclature is on the left):

$$C = u_1 \tag{a}$$

$$V_1+\Delta V_1 = s_{p_0+p}+\text{certain terms independent of } u_1 \tag{b}$$

$$\frac{f}{f_c} = 1+p \tag{c}$$

$$\mu = \varepsilon \tag{d}$$

from which it follows from equation (C1) and equation (57a) that

$$2A_2 = -S_{pu_1u_1} \tag{e}$$

$$3A_3 = \tfrac{1}{2}S_{u_1u_1u_1} \tag{f}$$

$$B_1 = S_{u_1\varepsilon}+O(p). \tag{g}$$

(C2)

Equation (11b) of [6], correcting the misprint in [6], is

$$u_1 = -\frac{2S_{pu_1u_1}}{S_{u_1u_1u_1}}p. \tag{C3}$$

Substitution from equation (C2) into equation (C3) gives the equilibrium equation for the slope of the initial post-buckling path for the ideal structure ($\varepsilon = 0$) as

$$C = \frac{2A_2}{3A_3}\left(\frac{f}{f_c}-1\right) \tag{C4}$$

which is the same as our equation (23).

For the slightly imperfect structure, $\varepsilon > 0$, equation (C1) defines an equilibrium path $p(u_1)$ along which the stationary value of p is p^* at $u_1 = u_1^*$:

$$u_1^* = -\frac{S_{pu_1u_1}}{S_{u_1u_1u_1}}p^*. \tag{C5}$$

Substitution for u_1^* in equation (C1) gives the stationary value for p^* {cf. equation (14) of [6]},

$$p^* = \pm\frac{[2S_{u_1u_1u_1}S_{u_1\varepsilon}\varepsilon]^{\frac{1}{2}}}{S_{pu_1u_1}} \tag{C6}$$

that is a necessary condition for snap-through. With proper choice of sign, equation (C6) is seen to be identical to equation (58a). The form taken by equation (58a), especially the dependence of the snap-through load on the square root of the imperfection for small imperfections, is to be expected from the general form of Roorda's result, equation (C6).

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Абстракт—Нелинейная задача несимметрического прощелкивания консольной колонны, закреп-, ленной на ее конце жесткой, наклоненной проволокой и нагруженной тамже горизонтальной силой, допускает точное решение, найденное уже раньше. Конструкция чувствительна к начальным неправильностям, если рассматривается случай, в котором роль неправильностей играют совместная обобщенная жесткость проволоки и центральная линия колонны. Теперь дается решение такой же самой задачи, используя общий метод расчета предложенный Койтером для почти закритического равновесия. Оказывается, что настоящее решение, касающиеся нагрузки выше критической точки в зависимости от прогиба, для "идеальной конструкции" является асимптотическим представлением соответствующего точного результата для затухающего малого прогиба. При положительном прогибе, приближенные значения для нагрузки по сравнению с точными являются меньшими в асимптотическом решении. Делается подобный вывод для нагрузки выпучивания в зависимости от амплитуды неправильностей для неправильно сначала мзготовленной конструкции /конечная обобщенная жесткость/.